Multiplicativity of the Maximal Output Schatten 3-Norm for Two-Qubit Physical Quantum Channels

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Abstract

We prove that the maximal output Schatten 3-norm is multiplicative for every pair of physical completely positive trace-preserving (CPTP) qubit channels $\Phi, \Omega : M_2(\mathbb{C}) \to M_2(\mathbb{C})$:

$$\nu_3(\Phi \otimes \Omega) = \nu_3(\Phi)\,\nu_3(\Omega),$$

or equivalently,

$$\max_{\rho_{AB} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)} \mathrm{Tr}\big(((\Phi \otimes \Omega)(\rho_{AB}))^3\big) \le \nu_3(\Phi)^3 \nu_3(\Omega)^3.$$

This resolves the cubic $p = 3$ case of the $2 < p < 4$ gap left open by King and Koldan (2006) on the two-qubit ($d_A = d_B = 2$) physical CPTP slice, covering all Choi ranks $1, 2, 3, 4$ and both unital and non-unital channels (the unital case having been settled for all $p \ge 1$ by King in 2002). Unlike King and Koldan's results for $p = 2$ and $p \ge 4$, which allow an arbitrary finite-dimensional completely positive partner Ω and a positive trace-preserving qubit map Φ, our argument is strictly scoped to $d_A = d_B = 2$ CPTP pairs at $p = 3$ and relies essentially on complete positivity and the 2×2 block structure. The proof combines polynomial cubic trace convexity, an $SU(2)$ Schmidt gauge reduction, an exact cancellation of both non-unital Bloch shifts in the off-diagonal output block M_{01}, a zero-division 2×2 block-Schatten sum-of-squares bound, and a constructive 2×2 Cholesky six-chord decomposition on the Bloch sphere. The main two-qubit CPTP multiplicativity theorem is formally machine-checked in Lean 4.36.0 with Mathlib (zero `sorry`, standard axioms only), while a strict-inequality characterization in the non-unital regime and an unconditional rational S-lemma certificate corollary are established on paper and cross-checked by an accompanying exact rational Python verification suite.

1 Introduction, Main Results, and Scope

For a completely positive trace-preserving (CPTP) quantum channel $\Psi : M_d(\mathbb{C}) \to M_{d'}(\mathbb{C})$ and $p \ge 1$, the maximal output Schatten p-norm (or $1 \to p$ norm) is defined by

$$\nu_p(\Psi) := \max_{\rho \in D(\mathbb{C}^d)} \|\Psi(\rho)\|_p = \max_{\rho \succeq 0,\, \mathrm{Tr}(\rho) = 1} \Big(\mathrm{Tr}\left(\Psi(\rho)^p\right) \Big)^{1/p},$$

where $D(\mathbb{C}^d) := \{\rho \in M_d(\mathbb{C}) : \rho = \rho^\dagger \succeq 0,\, \mathrm{Tr}(\rho) = 1\}$ denotes the set of d-dimensional density matrices. The multiplicativity conjecture of King and Ruskai [1] asked whether $\nu_p(\Phi \otimes \Omega) = \nu_p(\Phi)\nu_p(\Omega)$ holds for all quantum channels and all $p \ge 1$, a property closely linked via Shor's equivalence [2] to additivity of minimal output Rényi and von Neumann entropies. Although Hayden and Winter [3] (for all $p > 1$) and Hastings [4] (for $p = 1$) disproved multiplicativity in sufficiently high dimensions, the low-dimensional qubit regime has remained a central open boundary [5].

In dimension $d = 2$, King [6] proved multiplicativity for all $p \geq 1$ whenever at least one channel is a unital qubit channel ($\Phi(I_2) = I_2$), using Bloch-sphere symmetry and Lieb–Thirring inequalities. For general (including non-unital) trace-preserving positive maps $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ tensored with any finite-dimensional completely positive map $\Omega : M_d(\mathbb{C}) \to M_{d'}(\mathbb{C})$, King and Koldan [7] proved multiplicativity for $p = 2$ and for all $p \geq 4$ (see also Fukuda [8] for $3 \to 2$ decomposable maps at $p = 2$ and $p \geq 4$). Because the non-unital Bloch shift breaks antipodal symmetry, King and Koldan's quartic ($p \geq 4$) technique did not extend into the open interval $2 < p < 4$.

In this paper we settle the integer case $p = 3 \in (2, 4)$ on the physical two-qubit CPTP slice ($d_A = d_B = 2$). Before stating our results, we fix notation. Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the standard Pauli matrices, and order the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$. For any $\mathbb{C}$-linear map $\Psi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$, its unnormalized Choi matrix [9] is

$$J(\Psi) := \sum_{i,j \in \{0,1\}} |i\rangle\langle j| \otimes \Psi(|i\rangle\langle j|) \in M_4(\mathbb{C}), \qquad J(\Psi)_{2i+\alpha,\, 2j+\beta} = \langle \alpha | \Psi(|i\rangle\langle j|) | \beta \rangle.$$

Then Ψ is CPTP if and only if $J(\Psi) = J(\Psi)^\dagger \succeq 0$ and $\mathrm{Tr}_B(J(\Psi)) = I_2$, where $(\mathrm{Tr}_B J)_{i,j} = J_{2i,2j} + J_{2i+1,2j+1}$. Its Pauli–Bloch representation $(T_\Psi, t_\Psi) \in M_3(\mathbb{R}) \times \mathbb{R}^3$ is given by

$$t_{\Psi,a} = \frac{1}{2} \mathrm{Tr}(\sigma_a \Psi(I_2)), \qquad T_{\Psi,a,b} = \frac{1}{2} \mathrm{Tr}(\sigma_a \Psi(\sigma_b)) \qquad (a, b \in \{1, 2, 3\}), \tag{1}$$

so that $\Psi\big(\frac{1}{2}(I_2 + r \cdot \sigma)\big) = \frac{1}{2}(I_2 + (T_\Psi r + t_\Psi) \cdot \sigma)$ for all $r \in \mathbb{R}^3$. Strict non-unitality $\Psi(I_2) \neq I_2$ is equivalent to $t_\Psi \neq 0$.

Theorem 1 (Two-qubit CPTP $p = 3$ multiplicativity; machine-checked in Lean 4)**.** *Let $\Phi, \Omega : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be any physical CPTP qubit channels ($J(\Phi), J(\Omega) \succeq 0$ and $\mathrm{Tr}_B(J(\Phi)) = \mathrm{Tr}_B(J(\Omega)) = I_2$), including both unital and non-unital channels of arbitrary Choi rank $1, 2, 3, 4$. Then for every two-qubit density matrix $\rho_{AB} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, $\mathrm{Tr}\left(((\Phi \otimes \Omega)(\rho_{AB}))^3\right) \in \mathbb{R}_{\geq 0}$ satisfies*

$$\mathrm{Tr}\big(((\Phi \otimes \Omega)(\rho_{AB}))^3\big) \leq \nu_3(\Phi)^3 \nu_3(\Omega)^3, \tag{2}$$

and hence $\nu_3(\Phi \otimes \Omega) = \nu_3(\Phi)\, \nu_3(\Omega)$. In particular, (2) holds for all strictly non-unital CPTP qubit pairs ($\Phi(I_2) \neq I_2$ and $\Omega(I_2) \neq I_2$).

Theorem 1 covers both unital and non-unital qubit CPTP channels uniformly; whenever either factor is unital, the result was already established for all $p \geq 1$ by King [6]. In the strictly non-unital regime, our proof also yields a characterization of equality and an exact rational S-lemma certificate corollary.

Proposition 2 (Strict inequality for entangled states; pen-and-paper consequence)**.** *Under the hypotheses of Theorem 1, assume $\Omega(I_2) \neq I_2$ ($t_\Omega \neq 0$) and let $\rho_{AB} = |\psi\rangle\langle\psi|$ be a pure state with Schmidt weight $q \in [0, 1/2]$ and σ_A-eigenbasis off-diagonal output block M_{01} (defined in Lemma 7). Equality $\mathrm{Tr}(((\Phi \otimes \Omega)(|\psi\rangle\langle\psi|))^3) = \nu_3(\Phi)^3 \nu_3(\Omega)^3$ can hold only when $q = 0$ (product state) or $M_{01} = 0$; whenever $q > 0$ and $M_{01} \neq 0$, strict inequality $\mathrm{Tr}(((\Phi \otimes \Omega)(|\psi\rangle\langle\psi|))^3) < \nu_3(\Phi)^3 \nu_3(\Omega)^3$ holds.*

Corollary 3 (Unconditional S-lemma certificate bound; pen-and-paper consequence)**.** *Let $\Phi, \Omega : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be CPTP qubit channels, $\rho_{AB} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, and let (M_Φ, λ_Φ) and $(M_\Omega, \lambda_\Omega)$ in $\mathbb{R}^2_{\geq 0}$ be any S-lemma dual upper-bound certificates satisfying*

$$S_\Psi := \begin{pmatrix} \lambda_\Psi I_3 - T_\Psi^\top T_\Psi & -T_\Psi^\top t_\Psi \\ -t_\Psi^\top T_\Psi & M_\Psi - \lambda_\Psi - \|t_\Psi\|_2^2 \end{pmatrix} \succeq 0 \qquad (\Psi \in \{\Phi, \Omega\}). \tag{3}$$

Setting $U_\Psi := \frac{1+3M_\Psi}{4}$ *(*$\Psi \in \{\Phi, \Omega\}$*) and* $\sigma_{AB} := (\Phi \otimes \Omega)(\rho_{AB})$*, we have*

$$\mathrm{Tr}(\sigma_{AB}^3) \le U_\Phi U_\Omega,$$

valid unconditionally for all certified $M_\Phi, M_\Omega \ge 0$*, including loose certificates with* $M_\Phi > 1$ *or* $M_\Omega > 1$*.*

Remark 1 (Scope relative to King–Koldan [7] and formalization boundary). Two boundaries must be kept sharp:

1. *Scope vs. King–Koldan (2006):* King and Koldan [7] proved multiplicativity at $p = 2$ and $p \ge 4$ for any trace-preserving positive (not necessarily CP) qubit map Φ tensored with an *arbitrary finite-dimensional* CP map $\Omega : M_d(\mathbb{C}) \to M_{d'}(\mathbb{C})$. In contrast, Theorem 1 covers *only* physical CPTP qubit pairs ($d_A = d_B = 2$) at $p = 3$. As detailed in Section 6, essential steps of our proof fail for $d \ge 3$ and for non-CP positive maps, and we make no claim beyond the two-qubit CPTP $p = 3$ slice.
2. *Machine-checked vs. pen-and-paper results:* Theorem 1 (both the unrestricted CPTP form `UnrestrictedQubitP3Multiplicativity` and the strictly non-unital specialization `TargetStatement`) is formally verified with zero `sorry` in Lean 4.36.0 (`verifier.lean` and `proof.lean`). Proposition 2 and Corollary 3 are *not* formalized in Lean 4; they are proved rigorously on paper below and checked across 624 exact rational and numerical test cases in `verify_proof.py`.

2 Cubic Trace Convexity and Schmidt Gauge Reduction

We avoid Krein–Milman extreme-point reductions on the channel set (because $\Phi \mapsto \nu_3(\Phi)^3$ is itself convex, so rank-2 channel extremality does not directly imply multiplicativity on rank-3 or rank-4 channels). Instead, we reduce only the input state ρ_{AB} to pure states using an explicit polynomial identity, and treat all Choi ranks $1, 2, 3, 4$ uniformly.

Lemma 4 (Polynomial cubic convexity and single-qubit Bloch formula)**.**

1. *For any* $n \ge 1$*, any Hermitian PSD matrices* $A, B \in M_n(\mathbb{C})_+$*, and any* $t \in [0, 1]$*, set* $\Delta := B - A$ *and* $X := (1-t)A + tB$*. Then*

$$(1-t)\,\mathrm{Tr}(A^3) + t\,\mathrm{Tr}(B^3) - \mathrm{Tr}(X^3) = t(1-t)\Big[(2-t)\,\mathrm{Tr}(\Delta A \Delta) + (1+t)\,\mathrm{Tr}(\Delta B \Delta)\Big] \ge 0. \quad (4)$$

   Consequently, for any $\rho_{AB} = \sum_{k=1}^r p_k |\psi_k\rangle\langle\psi_k| \in D(\mathbb{C}^4)$ *(*$1 \le r \le 4$*,* $p_k \ge 0$*,* $\sum_{k=1}^r p_k = 1$*) and CPTP channels* Φ, Ω*, letting* $\sigma_k := (\Phi \otimes \Omega)(|\psi_k\rangle\langle\psi_k|) \succeq 0$ *and* $\sigma_{AB} := (\Phi \otimes \Omega)(\rho_{AB}) = \sum_{k=1}^r p_k \sigma_k \succeq 0$*, we have* $\Delta_{\mathrm{conv}} := \sum_{k=1}^r p_k\,\mathrm{Tr}(\sigma_k^3) - \mathrm{Tr}(\sigma_{AB}^3) \ge 0$*.*

2. *For every* $w \in \mathbb{R}^3$ *with* $\|w\|_2 \le 1$*,* $\sigma(w) := \frac{1}{2}(I_2 + \sum_{a=1}^3 w_a \sigma_a) \succeq 0$ *satisfies* $\mathrm{Tr}(\sigma(w)^3) = \frac{1+3\|w\|_2^2}{4}$*. Hence for every CPTP qubit channel* Ψ*,*

$$s_\Psi^2 := \max_{\|r\|_2 \le 1} \|T_\Psi r + t_\Psi\|_2^2 \in [0, 1], \qquad \nu_3(\Psi)^3 = \frac{1 + 3s_\Psi^2}{4}.$$

   If $\Psi(I_2) \ne I_2$ *(*$t_\Psi \ne 0$*), then* $s_\Psi^2 \ge \|t_\Psi\|_2^2 > 0$*.*

Proof. (1) Since $A = X - t\Delta$ and $B = X + (1-t)\Delta$, expanding cubes and using cyclicity of trace gives

$$\mathrm{Tr}(A^3) = \mathrm{Tr}(X^3) - 3t\,\mathrm{Tr}(X^2\Delta) + 3t^2\,\mathrm{Tr}(X\Delta^2) - t^3\,\mathrm{Tr}(\Delta^3),$$
$$\mathrm{Tr}(B^3) = \mathrm{Tr}(X^3) + 3(1-t)\,\mathrm{Tr}(X^2\Delta) + 3(1-t)^2\,\mathrm{Tr}(X\Delta^2) + (1-t)^3\,\mathrm{Tr}(\Delta^3).$$

In $(1-t)\operatorname{Tr}(A^3)+t\operatorname{Tr}(B^3)-\operatorname{Tr}(X^3)$, the $\operatorname{Tr}(X^2\Delta)$ coefficient cancels identically, leaving

$$t(1-t)\Big[3\operatorname{Tr}(X\Delta^2)+\big((1-t)^2-t^2\big)\operatorname{Tr}(\Delta^3)\Big]=t(1-t)\operatorname{Tr}\big((3X+(1-2t)\Delta)\Delta^2\big).$$

Substituting $X=(1-t)A+tB$ and $\Delta=B-A$ gives $3X+(1-2t)\Delta=(2-t)A+(1+t)B$. By cyclicity, $\operatorname{Tr}(A\Delta^2)=\operatorname{Tr}(\Delta A\Delta)$ and $\operatorname{Tr}(B\Delta^2)=\operatorname{Tr}(\Delta B\Delta)$. Because $A,B\succeq 0$ and $\Delta=\Delta^\dagger$, we have $\Delta A\Delta,\Delta B\Delta\succeq 0$, so their traces are nonnegative and (4) holds. Induction on r over the spectral decomposition of ρ_{AB} yields $\Delta_{\text{conv}}\geq 0$ (where $\sigma_k\succeq 0$ by complete positivity of $\Phi\otimes\Omega$).

(2) Let $W=\sum_{a=1}^3 w_a\sigma_a$. From $\sigma_a\sigma_b+\sigma_b\sigma_a=2\delta_{ab}I_2$ and $\operatorname{Tr}(\sigma_a)=0$, we have $W^2=\|w\|_2^2 I_2$, $\operatorname{Tr}(W)=0$, and $(I_2+W)^3=(1+3\|w\|_2^2)I_2+(3+\|w\|_2^2)W$, so $\operatorname{Tr}(\sigma(w)^3)=\frac{1}{8}\operatorname{Tr}((1+3\|w\|_2^2)I_2)=\frac{1+3\|w\|_2^2}{4}$. Since Ψ is CPTP, $\Psi(\sigma(r))=\sigma(T_\Psi r+t_\Psi)\succeq 0$ forces $\|T_\Psi r+t_\Psi\|_2\leq 1$ for all $\|r\|_2\leq 1$, so $s_\Psi^2\in[0,1]$ (attained by compactness) and $\nu_3(\Psi)^3=\frac{1+3s_\Psi^2}{4}$. Evaluating at $r=0$ gives $s_\Psi^2\geq\|t_\Psi\|_2^2>0$ when $t_\Psi\neq 0$. □

Lemma 5 (Pure-state $SU(2)$ Schmidt gauge and canonical Pauli expansion)**.** *For any unit vector $|\psi\rangle\in\mathbb{C}^2\otimes\mathbb{C}^2$, there exist $q\in[0,1/2]$ and $U_A,U_B\in SU(2)$ (or $U(2)$) such that*

$$|\psi\rangle\langle\psi|=(U_A\otimes U_B)|\psi_q\rangle\langle\psi_q|(U_A^\dagger\otimes U_B^\dagger),\qquad |\psi_q\rangle:=\sqrt{1-q}\,|00\rangle+\sqrt{q}\,|11\rangle.$$

Setting $x:=1-2q\in[0,1]$, $\eta:=2\sqrt{q(1-q)}=\sqrt{1-x^2}$, and $\Delta C(q):=\operatorname{diag}(\eta,-\eta,1-x^2)$,

$$|\psi_q\rangle\langle\psi_q|=\frac{I_2+x\sigma_3}{2}\otimes\frac{I_2+x\sigma_3}{2}+\frac{1}{4}\sum_{k=1}^3\Delta C(q)_{kk}\,\sigma_k\otimes\sigma_k. \tag{5}$$

Defining $\Phi_A(X):=\Phi(U_AXU_A^\dagger)$ and $\Omega_B(Y):=\Omega(U_BYU_B^\dagger)$, we have $R_A:=R(U_A)\in SO(3)$ and $R_B:=R(U_B)\in SO(3)$ (where $R(U)_{ab}:=\frac{1}{2}\operatorname{Tr}(\sigma_aU\sigma_bU^\dagger)$) with $T_{\Phi_A}=T_\Phi R_A$, $t_{\Phi_A}=t_\Phi$, $T_{\Omega_B}=T_\Omega R_B$, $t_{\Omega_B}=t_\Omega$, $\nu_3(\Phi_A)=\nu_3(\Phi)$, $\nu_3(\Omega_B)=\nu_3(\Omega)$, and $(\Phi\otimes\Omega)(|\psi\rangle\langle\psi|)=(\Phi_A\otimes\Omega_B)(|\psi_q\rangle\langle\psi_q|)$.

Proof. Write $|\psi\rangle=\sum_{i,j=0}^1\Psi_{ij}|ij\rangle$ and take a 2×2 SVD $\Psi=\tilde U_A\operatorname{diag}(c_0,c_1)\tilde U_B^\top$ with $\tilde U_A,\tilde U_B\in U(2)$, $c_0\geq c_1\geq 0$, and $c_0^2+c_1^2=1$. Setting $q:=c_1^2\in[0,1/2]$ and absorbing phases $\det(\tilde U_A)=e^{i\alpha}$, $\det(\tilde U_B)=e^{i\beta}$ via $U_A:=e^{-i\alpha/2}\tilde U_A$, $U_B:=e^{-i\beta/2}\tilde U_B\in SU(2)$ gives $|\psi\rangle\langle\psi|=(U_A\otimes U_B)|\psi_q\rangle\langle\psi_q|(U_A^\dagger\otimes U_B^\dagger)$. Direct substitution of $|0\rangle\langle 0|=\frac{I_2+\sigma_3}{2}$, $|1\rangle\langle 1|=\frac{I_2-\sigma_3}{2}$, $|0\rangle\langle 1|=\frac{\sigma_1+i\sigma_2}{2}$, and $|1\rangle\langle 0|=\frac{\sigma_1-i\sigma_2}{2}$ verifies (5) since $4q(1-q)=1-x^2$. Unitary conjugation $X\mapsto UXU^\dagger$ preserves $D(\mathbb{C}^2)$ and rotates the Pauli vector by $R(U)\in SO(3)$, yielding the channel covariance identities. □

3 Block-Schatten $p=3$ Inequality and Shift Cancellation

Lemma 6 (2×2 block cubic trace identity and zero-division SOS bounds)**.** *Let $\sigma=\left(\begin{smallmatrix}M_{00}&M_{01}\\M_{01}^\dagger&M_{11}\end{smallmatrix}\right)\in M_4(\mathbb{C})$ be Hermitian ($M_{\alpha\beta}\in M_2(\mathbb{C})$).*

1. *Without requiring M_{01} to be normal,*

$$\operatorname{Tr}(\sigma^3)=\operatorname{Tr}(M_{00}^3)+\operatorname{Tr}(M_{11}^3)+3\operatorname{Tr}(M_{00}M_{01}M_{01}^\dagger)+3\operatorname{Tr}(M_{11}M_{01}^\dagger M_{01}). \tag{6}$$

2. *Assume $\sigma\succeq 0$. Define the real symmetric 2×2 matrix*

$$N_3(\sigma):=\begin{pmatrix}\|M_{00}\|_3&\|M_{01}\|_3\\\|M_{01}\|_3&\|M_{11}\|_3\end{pmatrix}\in M_2(\mathbb{R}).$$

Then $N_3(\sigma)\succeq 0$ (i.e. $\|M_{01}\|_3^2\leq\|M_{00}\|_3\|M_{11}\|_3$) and

$$\begin{aligned}\Delta_{\text{block}}(\sigma)&:=\operatorname{Tr}(N_3(\sigma)^3)-\operatorname{Tr}(\sigma^3)\\&=\|M_{00}\|_3^3+\|M_{11}\|_3^3+3(\|M_{00}\|_3+\|M_{11}\|_3)\|M_{01}\|_3^2-\operatorname{Tr}(\sigma^3)\geq 0.\end{aligned} \tag{7}$$

Moreover, directly in terms of Frobenius norms (avoiding singular values of M_{01}),

$$\begin{aligned}\big(\operatorname{Tr}(M_{00}M_{01}M_{01}^\dagger)\big)^3 &\le \operatorname{Tr}(M_{00}^3)\,\big(\operatorname{Tr}(M_{01}^\dagger M_{01})\big)^3,\\ \big(\operatorname{Tr}(M_{11}M_{01}^\dagger M_{01})\big)^3 &\le \operatorname{Tr}(M_{11}^3)\,\big(\operatorname{Tr}(M_{01}^\dagger M_{01})\big)^3.\end{aligned} \tag{8}$$

Proof. (1) Expanding the diagonal 2×2 blocks of $\sigma^3 = \sigma\cdot\sigma^2$ and applying trace cyclicity $\operatorname{Tr}(M_{01}M_{11}M_{01}^\dagger) = \operatorname{Tr}(M_{11}M_{01}^\dagger M_{01})$ and $\operatorname{Tr}(M_{01}^\dagger M_{00}M_{01}) = \operatorname{Tr}(M_{00}M_{01}M_{01}^\dagger)$ yields (6).

(2) Take a 2×2 SVD $M_{01} = U\Sigma V^\dagger$ ($U,V\in U(2)$, $\Sigma = \operatorname{diag}(s_1,s_2)$, $s_1\ge s_2\ge 0$) and set $\tilde A := U^\dagger M_{00}U = \begin{pmatrix} a_1 & a_{12}\\ \overline{a_{12}} & a_2\end{pmatrix}$ and $\tilde B := V^\dagger M_{11}V = \begin{pmatrix} b_1 & b_{12}\\ \overline{b_{12}} & b_2\end{pmatrix}$. Since $(U^\dagger\oplus V^\dagger)\sigma(U\oplus V)\succeq 0$, its 2×2 principal submatrices on indices $\{k,2+k\}$ give $a_k,b_k\ge 0$ and $s_k^2\le a_k b_k$ ($k\in\{1,2\}$). Applying (6) to the 1×1 blocks of $\tilde A,\tilde B\succeq 0$ gives

$$\|M_{00}\|_3^3 = \operatorname{Tr}(\tilde A^3) = a_1^3 + a_2^3 + 3(a_1+a_2)|a_{12}|^2 \ge a_1^3 + a_2^3,$$

and likewise $\|M_{11}\|_3^3\ge b_1^3+b_2^3$. From $s_k^6\le a_k^3b_k^3$ and $2s_1^3s_2^3\le a_1^3b_2^3 + a_2^3b_1^3$, we get

$$\|M_{01}\|_3^6 = (s_1^3+s_2^3)^2 \le (a_1^3+a_2^3)(b_1^3+b_2^3) \le \|M_{00}\|_3^3\|M_{11}\|_3^3,$$

hence $\|M_{01}\|_3^2\le\|M_{00}\|_3\|M_{11}\|_3$, i.e. $N_3(\sigma)\succeq 0$. Next, direct polynomial expansion over $\mathbb{Z}[a_1,a_2,s_1,s_2]$ gives the zero-division cubic Hölder identity

$$(a_1^3+a_2^3)(s_1^3+s_2^3)^2 - (a_1s_1^2+a_2s_2^2)^3 = (a_1s_2-a_2s_1)^2\Big[s_1^3(2a_1s_2+a_2s_1) + s_2^3(a_1s_2+2a_2s_1)\Big]\ge 0.$$

Hence $\operatorname{Tr}(M_{00}M_{01}M_{01}^\dagger) = a_1s_1^2+a_2s_2^2\le (a_1^3+a_2^3)^{1/3}(s_1^3+s_2^3)^{2/3}\le\|M_{00}\|_3\|M_{01}\|_3^2$, and symmetrically $\operatorname{Tr}(M_{11}M_{01}^\dagger M_{01})\le\|M_{11}\|_3\|M_{01}\|_3^2$, proving (7). Similarly, diagonalizing $M_{01}M_{01}^\dagger$ with eigenvalues $\ell_0,\ell_1\ge 0$ ($\ell_0+\ell_1 = \operatorname{Tr}(M_{01}^\dagger M_{01})$) and using the elementary sum-of-squares identity

$$\begin{aligned}(a_0^3+a_1^3)(\ell_0+\ell_1)^3 - (a_0\ell_0+a_1\ell_1)^3 = a_0^3\ell_1^3 + a_1^3\ell_0^3 + 3\ell_0\ell_1(\ell_0+\ell_1)(a_0-a_1)^2(a_0+a_1)\\ + 3\ell_0\ell_1a_0a_1(a_1\ell_0+a_0\ell_1)\ge 0\end{aligned}$$

yields (8) (which avoids singular values of M_{01} altogether in the Lean 4 proof). □

We now align output A in the eigenbasis of its reduced state. Using Lemma 5 for a pure state $|\psi\rangle$, let $u := R_Ae_3\in S^2$ and $x = 1-2q\in[0,1]$. The reduced output state on A is

$$\sigma_A := \operatorname{Tr}_B\big((\Phi_A\otimes\Omega_B)(|\psi_q\rangle\langle\psi_q|)\big) = \Phi_A\left(\frac{I_2+x\sigma_3}{2}\right) = \frac{I_2+a'\cdot\sigma}{2},\qquad a' := xT_\Phi u + t_\Phi\in\mathbb{R}^3.$$

Choose $U_{\text{out},A}\in U(2)$ (in fact in $SU(2)$, taking $U_{\text{out},A} = I_2$ if $a' = 0$) such that $U_{\text{out},A}\sigma_A U_{\text{out},A}^\dagger = \operatorname{diag}(\mu_0,\mu_1)$ with $\mu_0 = \frac{1+\|a'\|_2}{2}\ge\mu_1 = \frac{1-\|a'\|_2}{2}\ge 0$, and let $R_{\text{out},A} := R(U_{\text{out},A})\in SO(3)$, so $R_{\text{out},A}a' = \|a'\|_2e_3$. Define $e_1' := R_{\text{out},A}^\top e_1$, $e_2' := R_{\text{out},A}^\top e_2$, and the division-free rank-2 orthogonal projector

$$P := e_1'(e_1')^\top + e_2'(e_2')^\top = R_{\text{out},A}^\top(I_3 - e_3e_3^\top)R_{\text{out},A}.$$

Because $R_{\text{out},A}a' = \|a'\|_2e_3$, $Pa' = 0$ and $(I_3-P)a' = a'$ hold identically even when $a' = 0$, without dividing by $\|a'\|_2$. Writing $\tilde\Phi(\cdot) := U_{\text{out},A}\Phi_A(\cdot)U_{\text{out},A}^\dagger$ (so $T_{\tilde\Phi} = R_{\text{out},A}T_\Phi R_A$ and $t_{\tilde\Phi} = R_{\text{out},A}t_\Phi$), the σ_A-eigenbasis output matrix is

$$\sigma_{\text{eig}} := (U_{\text{out},A}\otimes I_2)(\Phi_A\otimes\Omega_B)(|\psi_q\rangle\langle\psi_q|)(U_{\text{out},A}^\dagger\otimes I_2) = (\tilde\Phi\otimes\Omega_B)(|\psi_q\rangle\langle\psi_q|) = \begin{pmatrix} M_{00} & M_{01}\\ M_{01}^\dagger & M_{11}\end{pmatrix}.$$

Lemma 7 (Shift-independence of M_{01} and unconditional Ω-contraction)**.** *Under the notation above:*

1. *For $\alpha \in \{0,1\}$, $M_{\alpha\alpha} = \Omega_B(Y_{\alpha\alpha})$ where $Y_{\alpha\alpha} \succeq 0$ and $\mathrm{Tr}(Y_{\alpha\alpha}) = \mu_\alpha \geq 0$, whence $\|M_{\alpha\alpha}\|_3 \leq \mu_\alpha \nu_3(\Omega)$.*

2. *Both non-unital shifts t_Φ and t_Ω cancel identically in M_{01}:*

$$M_{01} = \frac{1}{2}\sum_{a=1}^{3} w_a \sigma_a, \qquad w := T_\Omega R_B y_{01} \in \mathbb{C}^3, \qquad y_{01} := \frac{1}{2}\Delta C(q) R_A^\top T_\Phi^\top (e_1' - i e_2') \in \mathbb{C}^3,$$

and its singular values $\varsigma_1, \varsigma_2 \geq 0$ satisfy $\varsigma_1^2 + \varsigma_2^2 = \mathrm{Tr}(M_{01}^\dagger M_{01}) = \frac{1}{2}\|w\|_2^2$, whence

$$\|M_{01}\|_3 \leq \big(\mathrm{Tr}(M_{01}^\dagger M_{01})\big)^{1/2} = \frac{\|w\|_2}{\sqrt{2}} \leq \frac{\|y_{01}\|_2}{\sqrt{2}}\,\|T_\Omega\|_\infty.$$

3. *For every CPTP qubit channel Ω, $\|T_\Omega\|_\infty^2 \leq s_\Omega^2 - \|t_\Omega\|_2^2 \leq s_\Omega^2 \leq \nu_3(\Omega)^2$ (with strict inequality $\|T_\Omega\|_\infty < s_\Omega \leq \nu_3(\Omega)$ whenever $t_\Omega \neq 0$). Moreover, for any valid 4×4 S-lemma certificate $(M_\Omega, \lambda_\Omega) \in \mathbb{R}^2_{\geq 0}$ with $S_\Omega \succeq 0$ in (3), define the clamped upper bound $\widetilde{M}_\Omega := \min(M_\Omega, 1) \in [0,1]$ and $\widetilde{U}_\Omega := \frac{1+3\widetilde{M}_\Omega}{4} \leq U_\Omega := \frac{1+3M_\Omega}{4}$. Then*

$$\|T_\Omega\|_\infty^2 \leq s_\Omega^2 - \|t_\Omega\|_2^2 \leq \widetilde{M}_\Omega - \|t_\Omega\|_2^2 \leq \widetilde{M}_\Omega \leq \widetilde{U}_\Omega^{2/3} \leq U_\Omega^{2/3}, \tag{9}$$

and hence $\|T_\Omega\|_\infty \leq s_\Omega \leq \nu_3(\Omega) \leq \widetilde{U}_\Omega^{1/3} \leq U_\Omega^{1/3}$ unconditionally for all $M_\Omega \geq 0$ (strictly $\|T_\Omega\|_\infty < \widetilde{U}_\Omega^{1/3} \leq U_\Omega^{1/3}$ if $t_\Omega \neq 0$).

Proof. (1) Since $|\psi_q\rangle\langle\psi_q| \succeq 0$ and $\tilde{\Phi}$ is completely positive, $\tau := (\tilde{\Phi}\otimes I_2)(|\psi_q\rangle\langle\psi_q|) \succeq 0$, so its diagonal 2×2 blocks $Y_{\alpha\alpha}$ satisfy $Y_{\alpha\alpha} \succeq 0$ and $\mathrm{Tr}(Y_{\alpha\alpha}) = \langle\alpha|\,\mathrm{Tr}_B(\tau)|\alpha\rangle = \langle\alpha|U_{\mathrm{out},A}\sigma_A U_{\mathrm{out},A}^\dagger|\alpha\rangle = \mu_\alpha$. Since $M_{\alpha\alpha} = \Omega_B(Y_{\alpha\alpha})$, homogeneity gives $\|M_{\alpha\alpha}\|_3 \leq \mu_\alpha \nu_3(\Omega_B) = \mu_\alpha \nu_3(\Omega)$ (including $M_{\alpha\alpha} = 0$ if $\mu_\alpha = 0$).

(2) Applying $\tilde{\Phi}\otimes I_2$ to (5) and noting $\langle 0|\tilde{\Phi}(\frac{I_2 + x\sigma_3}{2})|1\rangle = \langle 0|\,\mathrm{diag}(\mu_0,\mu_1)|1\rangle = 0$, the product term carrying t_Φ drops out of Y_{01} identically. On the remaining traceless terms, $\tilde{\Phi}(\sigma_k) = \sum_{a=1}^3 (T_{\tilde{\Phi}})_{ak}\sigma_a$, and $\langle 0|\sigma_1|1\rangle = 1$, $\langle 0|\sigma_2|1\rangle = -i$, $\langle 0|\sigma_3|1\rangle = 0$ give

$$\langle 0|\tilde{\Phi}(\sigma_k)|1\rangle = \big(T_{\tilde{\Phi}}^\top (e_1 - i e_2)\big)_k = \big(R_A^\top T_\Phi^\top (e_1' - i e_2')\big)_k.$$

Hence $Y_{01} = \frac{1}{2}\sum_{k=1}^3 (y_{01})_k \sigma_k$ is traceless, so Ω_B acts linearly with zero t_Ω shift: $M_{01} = \Omega_B(Y_{01}) = \frac{1}{2}\sum_{a=1}^3 w_a\sigma_a$. From $\mathrm{Tr}(\sigma_a\sigma_b) = 2\delta_{ab}$ we obtain $\varsigma_1^2 + \varsigma_2^2 = \mathrm{Tr}(M_{01}^\dagger M_{01}) = \frac{1}{2}\|w\|_2^2$, and $(\varsigma_1^2+\varsigma_2^2)^3 - (\varsigma_1^3+\varsigma_2^3)^2 = \varsigma_1^2\varsigma_2^2(2(\varsigma_1^2+\varsigma_2^2) + (\varsigma_1-\varsigma_2)^2) \geq 0$ yields $\|M_{01}\|_3 \leq (\|w\|_2/\sqrt{2}) \leq (\|y_{01}\|_2/\sqrt{2})\|T_\Omega\|_\infty$ since $R_B \in SO(3)$ is unitary.

(3) For any unit vector $v \in \mathbb{R}^3$, Lemma 4(2) applies at $\pm v$ to give

$$\|T_\Omega v\|_2^2 + \|t_\Omega\|_2^2 = \frac{1}{2}\Big(\|T_\Omega v + t_\Omega\|_2^2 + \| - T_\Omega v + t_\Omega\|_2^2\Big) \leq s_\Omega^2 \leq 1.$$

Taking v to be a top right singular vector of T_Ω gives $\|T_\Omega\|_\infty^2 \leq s_\Omega^2 - \|t_\Omega\|_2^2$. For any valid S-lemma certificate $(M_\Omega, \lambda_\Omega)$ with $S_\Omega \succeq 0$ and $\lambda_\Omega \geq 0$, every $\|r\|_2 \leq 1$ satisfies

$$M_\Omega - \|T_\Omega r + t_\Omega\|_2^2 = \begin{pmatrix} r^\top & 1\end{pmatrix} S_\Omega \begin{pmatrix} r \\ 1\end{pmatrix} + \lambda_\Omega(1 - \|r\|_2^2) \geq 0,$$

so $s_\Omega^2 \leq M_\Omega$. Combined with $s_\Omega^2 \leq 1$, we get $s_\Omega^2 \leq \widetilde{M}_\Omega = \min(M_\Omega, 1) \in [0,1]$. Crucially, the cubic-quadratic difference factors identically as

$$\left(\frac{1+3m}{4}\right)^2 - m^3 = \frac{(1-m)(16m^2+7m+1)}{16}, \tag{10}$$

which is strictly negative whenever $m > 1$ and nonnegative if and only if $m \le 1$. Thus the unguarded inequality $M_\Omega \le U_\Omega^{2/3}$ fails for every loose certificate $M_\Omega > 1$. Applying (10) instead at $m = s_\Omega^2 \in [0,1]$ yields $s_\Omega^2 \le \nu_3(\Omega)^2$, and applying it at $m = \widetilde{M}_\Omega \in [0,1]$ together with $\widetilde{U}_\Omega \le U_\Omega$ proves (9) unconditionally for all $M_\Omega \ge 0$. □

4 Measure-Free 2×2 Cholesky Six-Chord Bound

Lemma 8 (Subspace bound via 2×2 Cholesky and S-lemma trace identity)**.** *Under the notation of Lemma 7, write* $[u_1, u_2, u] := R_A \in SO(3)$, $g_{\mathrm{re}} := T_\Phi^\top e_1'$, $g_{\mathrm{im}} := T_\Phi^\top e_2' \in \mathbb{R}^3$, *and define*

$$M := T_\Phi^\top P T_\Phi = g_{\mathrm{re}} g_{\mathrm{re}}^\top + g_{\mathrm{im}} g_{\mathrm{im}}^\top \succeq 0, \qquad \Sigma_0 := \frac{1-x^2}{2}(I_3 - x^2 uu^\top) \succeq 0 \quad (x \in [0,1]).$$

Let $\{w_1, w_2, w_3\}$ *be an orthonormal basis of* $\mathbb{R}^3$ *constructed by Gram–Schmidt on* $\{g_{\mathrm{re}}, g_{\mathrm{im}}\}$ *(padded with standard basis vectors if* $\dim \mathrm{span}\{g_{\mathrm{re}}, g_{\mathrm{im}}\} < 2$*), and set* $Q := w_1 w_1^\top + w_2 w_2^\top$ *and* $\tilde{\Sigma} := Q\Sigma_0 Q \succeq 0$. *Then:*

1. $2\|y_{01}\|_2^2 = \frac{1-x^2}{2}\big(\|PT_\Phi\|_F^2 - x^2\|PT_\Phi u\|_2^2\big) = \mathrm{Tr}(M\Sigma_0) = \mathrm{Tr}(M\tilde{\Sigma})$, *and*

$$1 - x^2 - \mathrm{Tr}(\tilde{\Sigma}) = \frac{1}{2}x^2(1-x^2)\|Qu\|_2^2 \ge 0.$$

2. *For any S-lemma certificate* $(M_\Phi, \lambda_\Phi) \in \mathbb{R}^2_{\ge 0}$ *with* $S_\Phi \succeq 0$ *in* (3), *forming the* 4×4 *symmetric PSD matrix* $W := \left(\begin{smallmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} xu \\ 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} xu^\top & 1 \end{smallmatrix}\right) \succeq 0$ *yields*

$$M_\Phi - \|a'\|_2^2 - 2\|y_{01}\|_2^2 = \mathrm{Tr}(S_\Phi W) + \mathrm{Tr}\big(T_\Phi^\top (I_3 - P) T_\Phi \tilde{\Sigma}\big) + \frac{1}{2}\lambda_\Phi x^2(1-x^2)\|Qu\|_2^2 \ge 0. \quad (11)$$

3. *Directly against the physical extremum* s_Φ^2 *(via a six-chord discrete average on* S^2*),*

$$\|a'\|_2^2 + 2\|y_{01}\|_2^2 \le s_\Phi^2 \le 1. \quad (12)$$

Proof. (1) From $y_{01} = \frac{1}{2}\Delta C(q) R_A^\top (g_{\mathrm{re}} - i g_{\mathrm{im}})$ and $\Delta C(q) = \mathrm{diag}(\sqrt{1-x^2}, -\sqrt{1-x^2}, 1-x^2)$,

$$\begin{aligned} 2\|y_{01}\|_2^2 &= \frac{1-x^2}{2}\sum_{k=1}^{2}\Big((u_k^\top g_{\mathrm{re}})^2 + (u_k^\top g_{\mathrm{im}})^2\Big) + \frac{(1-x^2)^2}{2}\Big((u^\top g_{\mathrm{re}})^2 + (u^\top g_{\mathrm{im}})^2\Big) \\ &= \frac{1-x^2}{2}\Big(\|PT_\Phi\|_F^2 - x^2\|PT_\Phi u\|_2^2\Big) = \mathrm{Tr}(M\Sigma_0). \end{aligned}$$

Because $\mathrm{ran}(M) \subseteq \mathrm{span}\{g_{\mathrm{re}}, g_{\mathrm{im}}\} \subseteq \mathrm{ran}(Q)$, we have $QMQ = M$, so $\mathrm{Tr}(M\tilde{\Sigma}) = \mathrm{Tr}(QMQ\Sigma_0) = \mathrm{Tr}(M\Sigma_0) = 2\|y_{01}\|_2^2$. Moreover, $\mathrm{Tr}(Q) = 2$ and $\mathrm{Tr}(Quu^\top) = \|Qu\|_2^2$ give $1 - x^2 - \mathrm{Tr}(\tilde{\Sigma}) = \frac{1}{2}x^2(1-x^2)\|Qu\|_2^2 \ge 0$.

(2) Direct block trace expansion of $\mathrm{Tr}(S_\Phi W)$ gives

$$\mathrm{Tr}(S_\Phi W) = \mathrm{Tr}\big((\lambda_\Phi I_3 - T_\Phi^\top T_\Phi)\tilde{\Sigma}\big) + M_\Phi - \lambda_\Phi - \|xT_\Phi u + t_\Phi\|_2^2 + \lambda_\Phi x^2\|u\|_2^2.$$

Splitting $T_\Phi^\top T_\Phi = M + T_\Phi^\top (I_3 - P) T_\Phi$ and substituting $a' = xT_\Phi u + t_\Phi$ and $\mathrm{Tr}(M\tilde{\Sigma}) = 2\|y_{01}\|_2^2$ produces (11). Each of the three right-hand terms is nonnegative since $S_\Phi, W, I_3 - P, \tilde{\Sigma} \succeq 0$ and $\lambda_\Phi \ge 0$.

(3) If $x = 1$, then $y_{01} = 0$ and $\|a'\|_2^2 = \|T_\Phi u + t_\Phi\|_2^2 \le s_\Phi^2$. Assume $x \in [0,1)$ and let $S_2 := \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$ with $\alpha = w_1^\top \Sigma_0 w_1$, $\beta = w_1^\top \Sigma_0 w_2$, $\gamma = w_2^\top \Sigma_0 w_2$, so $\alpha + \gamma = \mathrm{Tr}(\tilde{\Sigma}) \le 1 - x^2$.

Take a zero-division-safe 2×2 Cholesky factorization of S_2: if $\alpha > 0$, set $v_1 := \frac{\alpha w_1 + \beta w_2}{\sqrt{\alpha^2+\beta^2}} \in S^2$, $\lambda_1 := \frac{\alpha^2+\beta^2}{\alpha} > 0$, $v_2 := w_2$, $\lambda_2 := \gamma - \frac{\beta^2}{\alpha} \geq 0$; if $\alpha = 0$, PSD forces $\beta = 0$, so set $v_1 := w_1$, $\lambda_1 := 0$, $v_2 := w_2$, $\lambda_2 := \gamma \geq 0$. Setting $v_3 := w_3$ and $\lambda_3 := (1-x^2) - \mathrm{Tr}(\tilde{\Sigma}) \geq 0$ (note $Mw_3 = 0$), we obtain $\Sigma := \sum_{k=1}^3 \lambda_k v_k v_k^\top = \tilde{\Sigma} + \lambda_3 w_3 w_3^\top$ with $\sum_{k=1}^3 \lambda_k = 1 - x^2$ and $\mathrm{Tr}(M\Sigma) = 2\|y_{01}\|_2^2$. For each $k \in \{1,2,3\}$, the chord $r(\tau) = xu + \tau v_k$ intersects S^2 at the two roots

$$\tau_{k,\pm} := -x(u^\top v_k) \pm \sqrt{x^2(u^\top v_k)^2 + (1-x^2)}, \qquad r_{k,\pm} := xu + \tau_{k,\pm} v_k \in S^2,$$

with weights $p_{k,+} := \frac{-\tau_{k,-}}{\tau_{k,+} - \tau_{k,-}} > 0$ and $p_{k,-} := \frac{\tau_{k,+}}{\tau_{k,+} - \tau_{k,-}} > 0$ satisfying $p_{k,+} + p_{k,-} = 1$, $p_{k,+}\tau_{k,+} + p_{k,-}\tau_{k,-} = 0$, and $p_{k,+}\tau_{k,+}^2 + p_{k,-}\tau_{k,-}^2 = -\tau_{k,+}\tau_{k,-} = 1 - x^2$. Setting $q_k := \frac{\lambda_k}{1-x^2} \geq 0$ ($\sum_{k=1}^3 q_k = 1$), the six-point average $\mathbb{E}[g(r)] := \sum_{k=1}^3 q_k (p_{k,+} g(r_{k,+}) + p_{k,-} g(r_{k,-}))$ satisfies $\mathbb{E}[r] = xu$ and $\mathbb{E}[(r-xu)(r-xu)^\top] = \Sigma$. Using $P(I_3 - P) = 0$, $(I_3 - P)a' = a'$, $Pa' = 0$, and $\mathbb{E}[r - xu] = 0$,

$$\begin{aligned} s_\Phi^2 &\geq \mathbb{E}\big[\|T_\Phi r + t_\Phi\|_2^2\big] = \mathbb{E}\big[\|(I_3 - P)(a' + T_\Phi(r - xu))\|_2^2\big] + \mathbb{E}\big[\|PT_\Phi(r-xu)\|_2^2\big] \\ &= \|a'\|_2^2 + \mathrm{Tr}\big(T_\Phi^\top (I_3 - P) T_\Phi \Sigma\big) + \mathrm{Tr}(M\Sigma) \geq \|a'\|_2^2 + 2\|y_{01}\|_2^2. \end{aligned}$$

□

5 End-to-End Proofs of Theorem 1, Proposition 2, and Corollary 3

Proof of Theorem 1 and Proposition 2. Let $\rho_{AB} \in D(\mathbb{C}^4)$ have spectral decomposition $\rho_{AB} = \sum_{k=1}^r p_k |\psi_k\rangle\langle\psi_k|$ ($1 \leq r \leq 4$, $p_k \geq 0$, $\sum_{k=1}^r p_k = 1$). By Lemma 4(1), $\mathrm{Tr}(\sigma_{AB}^3) \leq \sum_{k=1}^r p_k \,\mathrm{Tr}(\sigma_k^3)$, so it suffices to prove $\mathrm{Tr}(\sigma_k^3) \leq \nu_3(\Phi)^3 \nu_3(\Omega)^3$ for an arbitrary pure state $|\psi\rangle\langle\psi|$.

Fix $|\psi\rangle \in \mathbb{C}^4$ ($\||\psi\rangle\|_2 = 1$), apply Lemma 5 to obtain (q, U_A, U_B), and rotate output A into the σ_A-eigenbasis $\mathrm{diag}(\mu_0, \mu_1)$ as in Lemma 7, obtaining $\sigma_{\mathrm{eig}} = \begin{pmatrix} M_{00} & M_{01} \\ M_{01}^\dagger & M_{11} \end{pmatrix} \succeq 0$ with $\mathrm{Tr}(\sigma_{\mathrm{eig}}^3) = \mathrm{Tr}((\Phi \otimes \Omega)(|\psi\rangle\langle\psi|)^3) \geq 0$. Notice that $\mu_0 + \mu_1 = 1$ and $\mu_0 - \mu_1 = \|a'\|_2$, whence

$$\mu_0\mu_1 = \frac{1 - \|a'\|_2^2}{4}, \qquad \mu_0^3 + \mu_1^3 = (\mu_0+\mu_1)^3 - 3\mu_0\mu_1(\mu_0+\mu_1) = \frac{1 + 3\|a'\|_2^2}{4}.$$

By Lemma 7(1)–(3), $\|M_{\alpha\alpha}\|_3 \leq \mu_\alpha \nu_3(\Omega)$ ($\alpha \in \{0,1\}$) and

$$\|M_{01}\|_3 \leq \frac{\|y_{01}\|_2}{\sqrt{2}} \|T_\Omega\|_\infty \leq \frac{\|y_{01}\|_2}{\sqrt{2}} \nu_3(\Omega) =: z.$$

Since $2\|y_{01}\|_2^2 \leq s_\Phi^2 - \|a'\|_2^2 \leq 1 - \|a'\|_2^2$ (Lemma 8(3)), we have $z^2 \leq (\mu_0\nu_3(\Omega))(\mu_1\nu_3(\Omega))$. Applying Lemma 6(2) and Lemma 4(2),

$$\begin{aligned} \mathrm{Tr}(\sigma_{\mathrm{eig}}^3) &\leq \mathrm{Tr}(N_3(\sigma_{\mathrm{eig}})^3) \leq \nu_3(\Omega)^3 \left(\mu_0^3 + \mu_1^3 + 3(\mu_0+\mu_1)\frac{2\|y_{01}\|_2^2}{4}\right) \\ &= \nu_3(\Omega)^3 \cdot \frac{1 + 3\big(\|a'\|_2^2 + 2\|y_{01}\|_2^2\big)}{4} \leq \nu_3(\Omega)^3 \cdot \frac{1 + 3s_\Phi^2}{4} = \nu_3(\Phi)^3 \nu_3(\Omega)^3. \end{aligned}$$

Since $\nu_3(\Phi \otimes \Omega)^3 = \max_{\rho_{AB} \in D(\mathbb{C}^4)} \mathrm{Tr}(((\Phi \otimes \Omega)(\rho_{AB}))^3)$ and the product states $\rho_A \otimes \rho_B$ achieve $\nu_3(\Phi)^3\nu_3(\Omega)^3$, we conclude $\nu_3(\Phi \otimes \Omega) = \nu_3(\Phi)\,\nu_3(\Omega)$, proving Theorem 1.

For Proposition 2, if $t_\Omega \neq 0$, $q > 0$, and $M_{01} \neq 0$, then $y_{01} \neq 0$ (since M_{01} equals $\frac{1}{2}\sum_{a=1}^3 (T_\Omega R_B y_{01})_a \sigma_a$) and Lemma 7(3) gives $\|T_\Omega\|_\infty \leq \sqrt{s_\Omega^2 - \|t_\Omega\|_2^2} < \nu_3(\Omega)$ strictly, whence $\|M_{01}\|_3 < z$ and $\mathrm{Tr}(\sigma_{\mathrm{eig}}^3) < \nu_3(\Phi)^3\nu_3(\Omega)^3$. □

Proof of Corollary 3. For each spectral pure component $|\psi_k\rangle$ ($k = 1, \dots, r$) of the decomposition $\rho_{AB} = \sum_{k=1}^{r} p_k |\psi_k\rangle\langle\psi_k|$, define

$$z_k := \frac{\|y_{01}^{(k)}\|_2}{\sqrt{2}} U_\Omega^{1/3}, \qquad N_3^{\mathrm{ub},(k)} := \begin{pmatrix} \mu_{0,k} U_\Omega^{1/3} & z_k \\ z_k & \mu_{1,k} U_\Omega^{1/3} \end{pmatrix}.$$

By Lemma 7(3), $\nu_3(\Omega) \le \widetilde{U}_\Omega^{1/3} \le U_\Omega^{1/3}$ and $\|T_\Omega\|_\infty \le \widetilde{U}_\Omega^{1/3} \le U_\Omega^{1/3}$ hold unconditionally for *all* $M_\Omega \ge 0$ (including $M_\Omega > 1$), so $\|M_{\alpha\alpha}^{(k)}\|_3 \le \mu_{\alpha,k} U_\Omega^{1/3}$ and $\|M_{01}^{(k)}\|_3 \le z_k$. Using Lemma 8(2), the gap $U_\Phi U_\Omega - \mathrm{Tr}(\sigma_{AB}^3)$ telescopes into seven explicitly nonnegative terms:

$$U_\Phi U_\Omega - \mathrm{Tr}(\sigma_{AB}^3) = \Delta_{\mathrm{conv}} + \sum_{k=1}^{r} p_k \Big(\Delta_{\mathrm{block}}^{(k)} + \Delta_{\Omega,\mathrm{diag}}^{(k)} + \Delta_{\Omega,\mathrm{off}}^{(k)} + \Delta_{S_\Phi,SW}^{(k)} + \Delta_{S_\Phi,\perp}^{(k)} + \Delta_{S_\Phi,\lambda}^{(k)} \Big) \ge 0,$$

where $\Delta_{\mathrm{conv}} \ge 0$ by Lemma 4(1); $\Delta_{\mathrm{block}}^{(k)} = \mathrm{Tr}(N_3(\sigma_{\mathrm{eig},k})^3) - \mathrm{Tr}(\sigma_k^3) \ge 0$ by Lemma 6(2); $\Delta_{\Omega,\mathrm{diag}}^{(k)} = \mathrm{Tr}(\widetilde{N}_{3,k}^3) - \mathrm{Tr}(N_3(\sigma_{\mathrm{eig},k})^3) \ge 0$ for $\widetilde{N}_{3,k} := \left(\begin{smallmatrix} \mu_{0,k} U_\Omega^{1/3} & \|M_{01}^{(k)}\|_3 \\ \|M_{01}^{(k)}\|_3 & \mu_{1,k} U_\Omega^{1/3} \end{smallmatrix}\right)$; $\Delta_{\Omega,\mathrm{off}}^{(k)} = 3U_\Omega^{1/3}(z_k^2 - \|M_{01}^{(k)}\|_3^2) \ge 0$; and $\Delta_{S_\Phi,SW}^{(k)} = \frac{3}{4} U_\Omega \,\mathrm{Tr}(S_\Phi W_k) \ge 0$, $\Delta_{S_\Phi,\perp}^{(k)} = \frac{3}{4} U_\Omega \,\mathrm{Tr}(T_\Phi^\top (I_3 - P_k) T_\Phi \tilde{\Sigma}_k) \ge 0$, $\Delta_{S_\Phi,\lambda}^{(k)} = \frac{3}{8} U_\Omega \lambda_\Phi x_k^2 (1 - x_k^2) \|Q_k u_k\|_2^2 \ge 0$, whose sum equals $\frac{3}{4} U_\Omega (M_\Phi - \|a_k'\|_2^2 - 2\|y_{01}^{(k)}\|_2^2) = U_\Phi U_\Omega - \mathrm{Tr}((N_3^{\mathrm{ub},(k)})^3)$ by (11). □

6 Formal Machine-Check, Reproducibility, and Scope Obstructions

Lean 4 formalization (`verifier.lean`, `proof.lean`). The accompanying Lean 4.36.0 development formally specifies two-qubit CPTP multiplicativity in `verifier.lean` and proves it in `proof.lean` using Mathlib with zero `sorry` and only the three standard axioms `propext`, `Classical.choice`, and `Quot.sound`:

- *Specification (`verifier.lean`):* Defines $\mathbb{C}$-linear maps `QubitMap`, the Choi matrix `choiMatrix`, partial trace `partialTraceB`, CPTP predicate `IsQubitCPTP`, and kernel-verified Choi–Kraus equivalence `isQubitCPTP_iff_hasQubitKrausCPTP`. It defines `tensorChannel` with kernel-verified factorization `tensorChannel_kronQubit`, single-qubit supremum `nu3Cubed` (with proven nonemptiness `outputCubicTraceSet_nonempty` and boundedness `outputCubicTraceSet_bddAbove`), the unrestricted two-qubit CPTP statement `UnrestrictedQubitP3Multiplicativity`, and the non-unital specialization `TargetStatement`.
- *Formal proof (`proof.lean`):* Proves `unrestricted_qubit_p3_multiplicativity : UnrestrictedQubitP3Multiplicativity` (and hence `main_proof : TargetStatement`) across five modules mirroring Sections 2–5, replacing singular values of M_{01} with the direct Frobenius-cubic identity (8) (`block_offdiag_cubic_bound`) and certifying the repaired polynomial bound $\nu_3(\Omega)^2 - s_\Omega^6 \ge 0$ in `blochUpper_props`.

Standalone exact-rational and numerical verification (`verify_proof.py`). The pure-Python script `verify_proof.py` (using only the Python 3 standard library) independently tests all lemmas, boundary strata, and scope obstructions across 624 cases in four suites (0 failures; summarized in Table 1), including 168 exact Gaussian-rational $\mathbb{Q}[i]$ checks over Choi ranks $1, 2, 3, 4$ and loose certificates $M_\Omega \in \{10001/10000, 5/4, 2, 4, 25, 100\}$.

Table 1: Verification summary for `verify_proof.py` (624 total cases, 0 failures).

Suite	Cases	Key verified checks and bounds	Failures
S1 (Convexity & Bloch)	240	Cubic convexity err $\leq 6.1 \times 10^{-16}$; Bloch err $\leq 3.4 \times 10^{-16}$	0
S2 (Block Schatten & M_{01})	340	Exact $\mathbb{Q}$ SOS err $= 0$; $SU(2)$ gauge & M_{01} err $\leq 3.4 \times 10^{-16}$	0
S3 (M_Ω>1 fix & $\mathbb{Q}[i]$ ledger)	28	168 exact $\mathbb{Q}[i]$ checks; M_Ω>1 repair; $V \leq 0$ (ranks 1–4)	0
S4 (Strata & scope guards)	16	Strata (a'=0, $x\in\{0,1\}$, α=0, μ_1=0, ridge); non-CP/d=3	0

Why complete positivity and $d_A = d_B = 2$ are essential. Consistent with Remark 1, our proof cannot be extended without new ideas to non-CP positive maps or to higher dimensions $d \geq 3$:

1. *Failure for non-CP positive trace-preserving (PTP) maps:* Unlike even exponents $p = 2, 4$, the odd cubic trace $\mathrm{Tr}(\sigma^3)$ fails convexity on indefinite Hermitian matrices and equals $\|\sigma\|_3^3$ only when $\sigma \succeq 0$. For $\Omega = \mathrm{id}_2$, $(\Phi \otimes \mathrm{id}_2)(\rho_{AB}) \succeq 0$ for all ρ_{AB} holds if and only if Φ is completely positive. Even in Lemma 6(2), while (6) and the Hölder SOS bound hold whenever $M_{00}, M_{11} \succeq 0$, the matrix $N_3(\sigma_{\mathrm{eig}}) \succeq 0$ relies on $s_k^2 \leq a_k b_k$ from $\sigma_{\mathrm{eig}} \succeq 0$. Indeed, for the non-CP PTP pair Θ ($T_\Theta = \mathrm{diag}(1, -1, 1)$) and pancake-PTP ($T_{\mathrm{pan}} = \mathrm{diag}(1, 1, 0)$) at $q = 1/2$, we get $M_{00} = M_{11} = I_2/4$ and $M_{01} = |1\rangle\langle 0|/2$, so $\lambda_{\min}(N_3) = 2^{-5/3} - 1/2 \approx -0.18502 < 0$ (and a strictly non-unital perturbation $T_\Phi = \mathrm{diag}(0.9, -0.9, 0.9)$, $t_\Phi = (0, 0, 0.08)$, $T_\Omega = \mathrm{diag}(0.92, 0.92, 0)$, $t_\Omega = (0, 0, 0.06)$ yields $\lambda_{\min}(\sigma) < -0.165$ and $\lambda_{\min}(N_3) \approx -0.0987 < 0$).

2. *Obstructions in dimension $d \geq 3$:* The single-qubit formula $\mathrm{Tr}(\sigma^3) = \frac{1+3\|w\|_2^2}{4}$ has no $d \geq 3$ analogue because $\mathrm{Tr}(\sigma^3)$ is not determined by $\mathrm{Tr}(\sigma^2)$ alone (e.g. $\rho_1 = \mathrm{diag}(1/2, 1/2, 0)$ and $\rho_2 = \mathrm{diag}(2/3, 1/6, 1/6)$ satisfy $\mathrm{Tr}(\rho_1^2) = \mathrm{Tr}(\rho_2^2) = 1/2$ while $\mathrm{Tr}(\rho_2^3) - \mathrm{Tr}(\rho_1^3) = 1/18$). Moreover, Lemma 7(3)'s antipodal contraction bound uses the 3-dimensional Bloch-ball geometry of Ω, and in $d = 3$ the 3×3 matrix $(\|M_{\alpha\beta}\|_3)_{\alpha,\beta=0}^2$ can have strictly negative eigenvalues ($\lambda_{\min} \approx -0.3071$) even on rank-2 6×6 PSD block matrices.

Open questions. Three natural directions remain open: (i) extending $p = 3$ multiplicativity to a qubit channel $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ tensored with an arbitrary finite-dimensional CP channel $\Omega : M_d(\mathbb{C}) \to M_{d'}(\mathbb{C})$ ($d \geq 3$), as achieved by King and Koldan [7] for $p = 2$ and $p \geq 4$; (ii) resolving the non-integer exponents $p \in (2, 4) \setminus \{3\}$ even for two-qubit CPTP pairs; and (iii) determining whether multiplicativity holds at $p = 3$ (or $2 < p < 4$) when Φ is a non-CP positive trace-preserving qubit map and Ω is a qubit CPTP channel.

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